Indian Statistical Institute
M.Math II Year
Second Semester Mid Semester Examination, 2004-2005
Stochastic Processes-II

Time: 3 hrs   Date:02-03-05   Max. Marks : 100

1. Let $(M_t)_{t\geq 0}$ be a stochastic process such that each M_t takes values on the non-negative integers. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables. Assume that $(Y_1, Y_2, \ldots)$ and $(M_t)_{t\geq 0}$ are independent. Define

$$X_t = \sum_{j=1}^{M_t} Y_j \quad , t \geq 0.$$

a) If M_t and Y_t have finite mean show that X_t has finite mean and $E[X_t] = E(M_t)EY_1$. [7]

b) If M_t and Y_i are in L^2, show that

$$\text{Var } (X_t) = E(M_t) \text{ Var } Y_1 + (E(Y_1))^2 \text{ Var } (M_t).$$

[5]

c) If (M_t) is a Poisson process find the covariance function of (X_t). [8]

2. Let (X_t) and (Y_t) be two mean zero L^2-processes on $[a,\ b]$. Let $K(s,t) = EX_s\overline{Y_t}$. Let $K(s,t)$ be continuous on $[a,\ b] \times [a,\ b]$. Let the covariance functions of (X_t) and (Y_t) be continuous on $[a,\ b]$.

a) If f and g are continuous functions on $[a,\ b]$ show that

$$E\left(\int_a^b f(s)X_s\, ds\right)\left(\overline{\int_a^b g(s)Y_s\, ds}\right) = \int_a^b\int_a^b f(s)\overline{g(t)}K(s,t)\, ds\, dt$$

[6]

b) If h is a continuous function on $[a,\ b]$, show that

$$EX_s\left(\overline{\int_a^b h(t)Y_t dt}\right) = \int_a^b K(s,t)\overline{h(t)}\, dt.$$

[5]

3. If λ_n , $n = 1, 2, \ldots$ are the non-zero eigenvalues of the integral equation $\int_a^b K(s,t)e(t)\ dt = \lambda\ e(s) \quad a \leq s \leq b$, where K is the continuous covariance function of a stationary L^2-process show that $\sum_{n=1}^{\infty} \lambda_n = c \cdot (b-a)$ for some positive constant c. [8]

4. Let $(X_t)_{t \in T}$ be a mean zero, stationary L^2-process with continuous covariance function. Let $H = L^2\{X_t,\ t \in T\}$. For $t \in T$, let $T_t X_s = X_{s+t}$.

   a) Show that T_t extends as a linear operator $T_t : H \to H$ which is unitary i.e., $TT^* = T^*T = Id$. [7]

   b) If Z is the spectral measure of (X_t) show that

$$T_t \int_S \phi(\lambda) dZ(\lambda) = \int_S e^{it\lambda} \phi(\lambda) dZ(\lambda)$$

[8]

5. Let $\{X_n\}$ be an auto regressive scheme with $\sum_{j=0}^{\infty} b_j X_{n-j} = W(n)$, where $\{W_n\}$ is white noise. Let $B(z) = \sum_{j=0}^{\infty} b_j z^j$ and suppose that the series converges uniformly in $\{z : |z| < R\},\ R > 1$ and has no zeroes on $\{z : |z| \leq 1\}$

   a) If $m < n$, show that

$$E[X(m)\overline{W(n)}] = \frac{1}{2\pi i} \int_{|z|=1} \frac{z^{n-m-1}}{B(z)}\ dz$$

   where the path of integration is counter clockwise. [7]

   b) Show that $X(m) \perp W(n)$ if $m < n$. [4]

6. Let $\{f_n(\lambda)\}$ be a uniformly bounded sequence of spectral densities such that $f_n(\lambda) \to f_0 > 0 \ \ \forall \lambda$. Show that $\exists\ L^2$- processes $\{W_n(t),\ -\infty < t < \infty\}$ such that

   a) $\forall\ g \in L^2(I\!R)$,

$$W(g) := \lim_{n \to \infty} \int_{-\infty}^{\infty} g(t) W_n(t)\ dt$$

exits in $L^2(\Omega, \mathcal{F}, P)$ [7]

b) $\forall\ g, h \in L^2(I\!\!R)$,

$$E[W(g)\overline{W(h)}] = 2\pi f_0 \int_{-\infty}^{\infty} g(t)\overline{h(t)}dt$$

[8]

7. Let $(B_t)_{t \in I\!\!R}$ be the extended Brownian motion. For $\lambda_1 < \lambda_2$ define

$$Z(\lambda_1,\ \lambda_2] = B_{\lambda_2} - B_{\lambda_1}$$

a) For $t_1 < t_2$ define

$$\hat{Z}(t_1, t_2] = \int_{I\!\!R} \frac{e^{i\lambda t_2} - e^{i\lambda t_1}}{i\lambda}\ dZ(\lambda)$$

Show that $\hat{Z}(t_1,\ t_2]$ is a well defined L^2- random variable. [5]

b) Show that if $-\infty < t_1 < t_2 \leq t_3 < t_4 < \infty$ then

$$E\hat{Z}(t_1,\ t_2]\overline{\hat{Z}(t_3,\ t_4]} = 2\pi\lambda(t_1,\ t_2] \cap (t_3,\ t_4]$$

where λ is Lebesgue measure on $I\!\!R$. [5]

c) Show that if $\phi \in L^2(d\lambda)$ then

$$\int_{-\infty}^{\infty} \phi(t)d\hat{Z}(t) = \int_{-\infty}^{\infty} \hat{\phi}(-\lambda)dZ(\lambda)$$

[10]